For office use only	Team Control Number	For office use only
T1 __________	**2002129**	F1 __________
T2 __________		F2 __________
T3 __________	Problem Chosen	F3 __________
T4 __________	**B**	F4 __________

2020
MCM/ICM
Summary Sheet

Method Analysis for Building the Longest Lasting Sandcastles

Summary

In this paper, we propose that an airfoil-like streamlined body can last the longest from the waves. We have figured out a Smoothed Particle Hydrodynamics(SPH) method to analyze the motion of sand and liquid. We consider the liquid as a compose of particles and analyze their motion respectively. Navier-Stokes momentum equation is applied to determine the accelaration of liquid particles. Then Biesel transfer function is considered to generate waves. Using the above methods, the motion of fluid can be simulated verisimilarly.

Moreover, we use porous flow simulation to resolve the flowing of liquind inside the sand particles. To determine whether a sand particle will yield under the influence of fluid, we utilize the Drucker-Prager(DP) model to analyze. This model depends highly on the input value of internal friction angle and cohesion pressure. We then define a new variable, proportion of potential energy loss, to indicate the shape change of the geometric foundation. Comparing with other four geometric shapes, we claim that airfoil-like streamlined body performs the best withstanding the waves.

As for the second question, we apply empirical formulas based on experiment results to obtain the function between moisture content and internal friction angle, cohesion pressure, as well as wet density of the sand. Since the SPH model depends on above three variables input, we choose several feature points of moisture content and insert their resulting calculation into previous SPH model and find that the best moisture content leading to the least energy loss is 14%.

Rainfall is simulated similarly as question 1, but changes the birth place of water from left sides of the foundation to the air above. We find that airfoil-like streamlined body is not the best geometric shape at this situation. Cube and triangular prism act best this time.

Furthermore, we come up with another three strategies to build a more stable sandcastle, including digging a drainage ditch, building a wall, and choosing a steep position. We also analyze the sensitivity of our model and comment on our advantages, limitations, and possible improvement. Finally, based on all of our calculation and simulation, we write an article to help those non-technical readers build a better sandcastle during their vacation on the beach.

Keywords: SPH; Streamlined Body; Biesel Transfer Functions; Porous Flow Simulation; Moisture Content; C++; Visio; ParaView; UGNX.

Method Analysis for Building the Longest Lasting Sandcastles

March 10, 2020

Contents

1 Introduction

1.1 Problem Overview

Sandcastles appear in every recreational sandy ocean beaches around the world. During the vacation on the seashore, travelers tend to make fully use of their imagination to create the incredible artworks. With simple tools and small skills, one can succeed to build complicated and enormous sandcastles together with their delicate features only using ordinary wetted sand.

Nevertheless, continual ocean waves and wising tides can easily invade and finally destroy the vivid sandcastles. Consequently, coming up a specific method to reduce the influence of waves and tides is of great importance. It is necessary for us to figure out the best 3-dimensional geometric shape which can be used to build the foundation of a sandcastle that withstand the assault of the wave best.

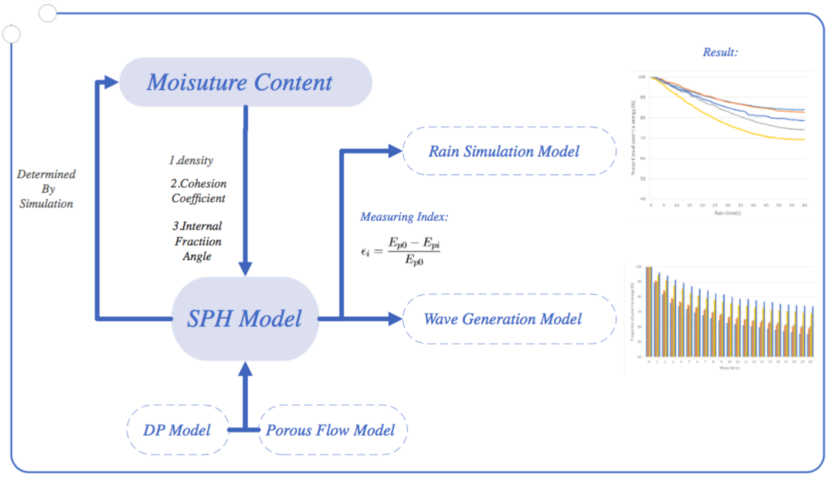
1.2 Background

Similar works to analyze the interaction between sand and liquid has been done. The motion of waves and tides can be solved by treating liquid as a set of particles according to Wei-Chun Teng [1], and then apply the method of SPH model. Using Navier-Stokes momentum equation and Smoothing Kernel Function given by Muller[2] and Monaghan J[3] 's research, the motion of liquid particles could be traced. Zhu and Bridson[4] provide a method to utilize the similar SPH model to analyze the float of solid, especially the motion of sand.

As for the best moisture content for sand, Shayea [5] figure out that there exist a limit of water content where the cohesion coefficient of sand reaches its maximum. Zhang[6] made experiments and successfully obtained the detailed relation function between the two quantities. Moreover, Schwarze's group[7] put forward a microcosmic explanation to examine the relation between volume of water and their capillary force.

With the help of their research, we will first construct a mathematical model and give the best geometric foundation for sandcastles, and then determine the optimal sand-to-water mixture proportion that gives the most stable castle foundation. Later we will determine the influence of rain on the sandcastles and give some useful strategies to make our sandcastles stronger. We will also write an article to help those non-technical readers build their powerful sandcastles.

1.3 Flow Diagram



2 Geometric Foundation Design

In this section, we propose our designed 3-dimensional geometric foundation that will last the longest period of time on the seashore, as shown in Figure 2.1. We call it Airfoil-like Streamlined Body.

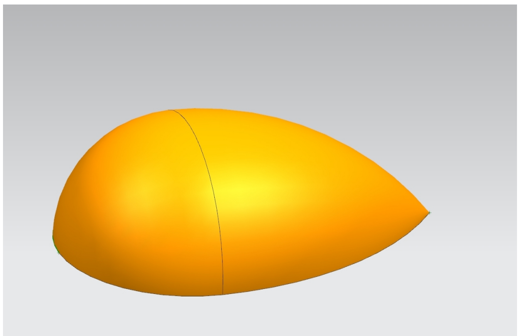


Figure 2.1: Airfoil-like Streamlined Body

This streamline body model we designed is rotated by a two-dimensional plane structure, as shown in Figure 2.2. The plane figure mainly consists of two arcs, and their centers are indicated in the figure. These two arcs rotated around the central axis for 180 degrees and form the 3-dimensional geometric foundation which is shown in Figure 2.1. The left part of this model is a quadrant, forming quarter ball. The center of the right ball is on the extension cord of the left circle's radius, as indicated below. The detailed parameters for this shape are recorded in Table 1 below.

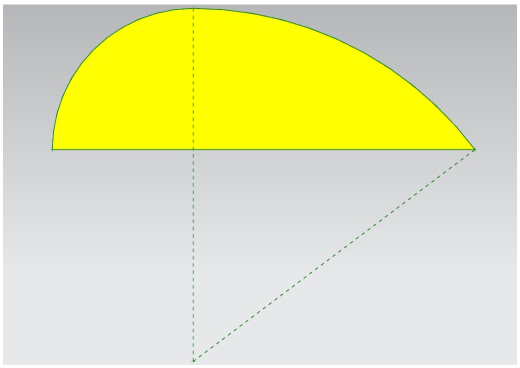


Figure 2.2: Two-Dimensional Plane Swtructure

	Arc 1	**Arc 2**
Diameter	0.400	1.000
Arc Length	0.314	0.464
Origin Angel	90.000	36.870
Final Angel	180.000	90.000
Central Point	(0.000,0.000,0.000)	(0.000,-0.300,0.000)
Origin Point	(0.000,0.200,0.000)	(0.400,0.000,0.000)
Final Point	(-0.200,0.000,0.000)	(0.000,0.200,0.000)

Table 1: Detailed Parameters for designed Model(Unit:m)

To help analyze the sensitivity of this model, we define a new parameter κ, which denotes the radius ratio of the two circles. We have $\kappa = \dfrac{R_2}{R_1}$, where R_1 is the radius of left arc and R_2 is the radius of the right arc. This is the only parameter that can influence the geometric shape of the model. The influence of this parameter will be discussed later.

3 Model

3.1 Notations and Symbols

VARIABLES	DEFINITION
κ	Radius Ratio
ρ_i	The Density of Particles
$\vec{u}$	The Velocity of Particles
m_i	The Mass of Particles
P_i	The Pressure among Particles
$W(\vec{r}-\vec{r_j},h)$	Smoothing Kernel Function
E_{pi}	The Total Potential Energy of the Foundation.
ϵ_i	The Proportion of Energy Loss
Φ_i	Porosity of Porous Particles
S_i	Saturation of Porous Particles
K_i	Permeability
P_i^C	Capillary Potential
P_i^p	Capillary Pressure
v_{pi}	Pore Velocity
θ	Internal Friction Angle of Sand
ρ_d	Dry Density of Sand
c	Cohesion Pressure inside the Sand
w	Moisture Content

Table 2: Notations and Symbols

3.2 General Assumptions

1. **The waves and tides are pure liquid that could be analyzed as flowing particles.**
   This is the foundamental assumption of SPH model. Sand itself is a combination of particles and is easy to understand. The most crucial part is treating fluid as a set of particles. In this way, we can easily apply motion equations on seperate particles and predict their future state.

2. **The waves and tides cannot penetrate into the ground sand.**
   This assumption aims for avoiding unnecessary complexity. We assume that there is no interaction with waves and ground, all we should consider is the collision between liquid particles and the sand foundation. This assumption can simplify our calculation to a great extend.

3. **The volume of sand will not change if we add small amount of water to it.**
   This is a crucial assumption used for calculating the relation between moisture content and wet density of sand. Since when the amount of adding water is small, the vary of volme is neglectable. Hence we think that this assumption is reasonable.

3.3 Model Analysis

There are mainly two perspectives to analyze the motion of liquid. One is the Euler method, which examine liquid in a fixed coordinate system. This is a traditional method to analyze fluid. Which requires the creation of grid cells. The other is Lagrange method, which regard liquid as flowing units.

Hence we also have proposed two methods to calculate the motion of liquid and sand. Our first designed model is cellular automata(CA), based on the concept of Euler method. We simplify the entire 3-dimensional space as an array of small hexahedron network, and consider liquid moving among these hexahedrons. However, this model restrict the motion of fluid to the exchange of grids, which is difficult to accurately simulate the dynamic motion equation of fluid under complex mechanics. Once we analyze the force, the whole system will become incredibly complex. Moreover, this system can not simulate the permeation of liquid into the sand.

Consequently, a better model, Smoothed Particle Hydrodynamics(SPH), is put forward, based on the concept of Lagrange method. The principal of this system is considering fluid as a set of motional particles. We apply Newtonian Mechanical Equations to a single particle and predict the motion state of all the particles in the next time node. With the help of this model, we can analyze and predict the detailed motion of both liquid and sand in a simple and accurate way. The advantages of our system will be discussed in details later.

3.4 Basic Model

3.4.1 Smoothed Particle Hydrodynamics(SPH) Model

The crucial part of our model is to simulate the motion of waves and tides. Hence we apply Smoothed Particle Hydrodynamics(SPH) Model in order to determine the numerical solution of the water dynamics equations.[1][8]

SPH model is a typical Lagrange method to analyze fluid dynamics. We consider the fluid as a set of particles which have a significant interaction with each other. Apply Newton's second law and replace mass with density, we have

$$\rho\vec{a} = \vec{F} \tag{1}$$

with the dimension of force F equals $MT^{-2}L^{-2}$. Force applied on a single fluid particle consists of three parts

$$\vec{F} = \vec{F}^{external} + \vec{F}^{pressure} + \vec{F}^{viscosity} \tag{2}$$

$\vec{F}^{external} = \rho\vec{g}$ is external force which composed of gravity in this situation. $\vec{F}^{pressure} = -\nabla P$ is the force resulting from the pressure difference inside the fluid, which equals the gradient of pressure field. $\vec{F}^{viscosity} = \mu\nabla^2\vec{u}$ is the force resulting from the velocity difference among the fluid particles

Using above equations, we have the Navier-Stokes momentum equation [2]

$$\frac{D\vec{u}}{Dt} = \vec{g} - \frac{\nabla P}{\rho} + \frac{\mu\nabla^2\vec{u}}{\rho} \tag{3}$$

The core function of SPH model is[3]

$$A(\vec{r}) = \sum_j A_j \frac{m_j}{\rho_j} W(\vec{r} - \vec{r_j}, h) \tag{4}$$

A_j is the property of fluid we want to examine, m_j and ρ_j are mass of the surrounding particles, and W is a Smoothing Kernel Function, with $\vec{r}$ is the position of the particles and h is the support radius.

Here we use the density ρ to replace A, obtaining

$$\rho_i = \sum_j m_j W(\vec{r} - \vec{r_j}, h) \tag{5}$$

The pressure can be calculated using Gas State Equation, which is

$$P_i = k(\frac{\rho_i}{\rho} - 1) \tag{6}$$

where ρ is physical fluid density and k is a constant. Hence detailed calculations for $\vec{F}^{pressure}$ and $\vec{F}^{viscosity}$ are given by

$$\vec{F}_{ij}^{pressure} = -V_i V_j (P_i + P_j)(\nabla W(r_{ij}, r_i) + \nabla W(r_{ij}, r_j))/2 \tag{7}$$

$$\vec{F}_{ij}^{viscosity} = \mu V_i V_j (u_j - u_i)(\nabla^2 W(r_{ij}, r_i) + \nabla^2 W(r_{ij}, r_j))/2 \tag{8}$$

3.4.2 Wave Generation

We apply Biesel transfer functions [Biesel and Suquet, 1951] to express first order wave.[10]

$$\eta(x,t) = \frac{H}{2} cos(\omega t - kx + \delta) \tag{9}$$

Biesel transfer functions proposes the relation between wave amplitude and wave maker's displacement. S_0 denotes the piston stroke, from which the time series of the piston movement is given by

$$e_1(t) = \frac{S_0}{2} sin(\omega t + \delta) \tag{10}$$

Hence the wave height H is given by

$$H = S_0 \frac{2 sinh^2(kd)}{sinh(kd) cosh(kd) + kd} \tag{11}$$

3.4.3 Porous Flow Simulation

Lenaerts' article[9] provides another useful proposal, porous flow simulation, to analyze the interaction between sand and liquid, which can also be taken into consideration. Sand particles are regarded as porous, containing a lot of holes, as shown in Figure 3.1 below.

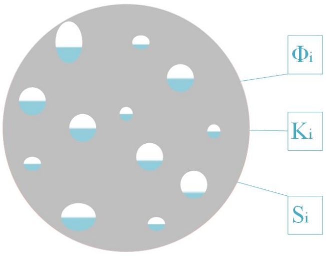


Figure 3.1: Porous Particle Model

We define porosity Φ_i as the void space per unit volume for particle p_i, and hence $\Phi_i V_i$ is the volume of void space. m_{pi} is the absorbed liquid mass. K_i is the permeability, which is a constant. Then we define particle's saturation S_i as

$$S_i = \frac{m_{pi}}{\rho^{fluid} \Phi_i V_i} \tag{12}$$

We can also find the relation between density and porosity as

$$\Phi_i \rho_i^C = \Phi_0 \rho_0^C = Constant \tag{13}$$

We define $P^C = k^C(1 - S_i)^\alpha$ as capillary potential, where k_C and $\alpha(0 < \alpha < 1)$ are constants. SPH model is applied again to do the calculation

$$\nabla P_i^C = \sum_j V_j P_j^C \nabla W(r_j - r_i, h) \tag{14}$$

Capillary pressure(P^p) is the factor pushing liquid inside porous particles, which can be calculated as

$$P_i^p = k^p S_i((\frac{\rho_0}{\rho_i})^{S\gamma} - 1) \tag{15}$$

According to Darcy's law[11], pore velocity can be calculated as

$$v_{pi} = -\frac{K_i}{\Phi_{i\mu}}(\nabla P_i^p - \nabla P_i^C - \rho g) \tag{16}$$

We find that this equation has many similarities as Equation (3). Using this method, we can easily track the motion of fluid inside sand particles.

3.4.4 Interaction between Liquid and Sand(DP Model)

Drucker-Prager(DP) model is applied to predict the yielding point of sand particles according to [10]. We have two parameters a and b writing as:

$$a = -\frac{2\sqrt{3}sin(\theta)}{3 - sin(\theta)} \tag{17}$$

$$b = \frac{2\sqrt{3}cos(\theta)}{3 - sin(\theta)}c \tag{18}$$

where θ is the internal friction angle, and c is the cohesion pressure.

Yielding of sand particles will not happen unless the following equation is satisfied:

$$b - aP < 0 \tag{19}$$

Where P is the pressure fluid applied on the sand particles. According to this model, we find that whether a sand particle will move or not depends highly on the internal friction angle and cohesion pressure, and these two parameters have a direct relation with the moisture content of sand. The influence of moisture content will be discussed in problem 2. As for problem 1, 3, and 4, we will choose the best value of them, which is $\theta = 29.0691^o$, $c = 29.9202kPa$, $\rho_i = 1.7727g/cm^3$.

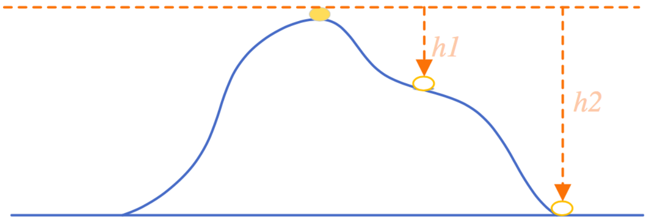
3.4.5 Measurement Index

To determine how our designed foundation is influenced by the waves and tides, similarly we treat sand as a set of particles and analyze their motion. We set a new variable E_{p0} to determine the origin total potential energy of the sandcastle foundation. When waves and tides erode the foundation, some sand will be washed away with the current or fall down from their origin position, so that the total potential will decrease. We note the potential energy of the sandcastle after i_{th} wave as E_{pi}, and the proportion of reduction of energy as ϵ_i, while

$$\epsilon_i = \frac{E_{p0} - E_{pi}}{E_{p0}} \tag{20}$$

Although we are using same amount of sand to construct the foundation, the origin potential energy E_{p0} will vary as we choose different geometric shape. However, ϵ_i can be a proper variable to determine the extent to which the foundation is eroded. Smaller ϵ_i represents more stable sandcastle foundation that will last longer by the sea.

As shown in Figure 3.2, considering a single sand particle on the foundation, if it is not influenced by the liquid particle, its position will remain the same and its potential will not change. If this particle slides down a little bit, from its origin position to to a new place h_1 lower than the origin, its potential will get smaller. If the particle flashes away with liquid or slides down to the ground h_2 lower than origin, its potential becomes zero. Since $h_1 < h_2$, the second condition indicates a worse case. Hence the energy loss proportion ϵ_i is a desired physical quantity which can quantitatively measure the degree of erosion of our designed foundation.



Since mgh1<mgh2, when the particle drop height h2,
we consider it a worse case

Figure 3.2: Potential Energy Loss

3.4.6 Program Implementation and Visualization

Based on our SPH model, we can simulate the dynamics process of wave and sand. We apply *C++* to do coding and *ParaView* to visualize the graph. The main pseudocode is listed below:

```
Algorithm 1: SPH Model for our system
Input: The origin position and speed(equals 0) of sand and liquid particles; The cohesion coefficient
    of the phase.
Output: Figure showing the state of sand foundation after $i_{th}$ collision; Energy Loss Proportion $\epsilon_i$.
1: for each particle $i$
2:     calculate $\rho_i$ and $P_i$ using Equation (5) and (6);
3:     calculate $\vec{F}^{external}$, $\vec{F}^{pressure}$, and $\vec{F}^{viscosity}$ using Equation (7) and (8);
4:     calculate $D\vec{u_i}/Dt$ using Navier-Stokes momentum equation;
5:     calculate $\vec{r_i}$ and $\vec{u_i}$ at next time node;
6: end for
7: use $\vec{r_i}$ and $\vec{u_i}$ as a new input and repeat procedure 1 until terminal time;
8: calculate $E_p$ and $\epsilon$;
9: return $\epsilon$;
```

Figure 3.3 indicate a sample of our simulation. Blue particles simulate waves and tides, gray particles simulates sand foundation, and yellow particles simulates ground beach. Based on our assumption, blue and yellow particles will not influence each other. Due to the effect of gravity, liquid will flow forward spontaneously and finally reach the position of sand foundation. Sand and liquid particles interact with each other as discussed in Section 3.4. Finally, some sand particles will move from where it was before, and we can calculate the potential energy loss proportion as an stability indicator of our designed foundation.

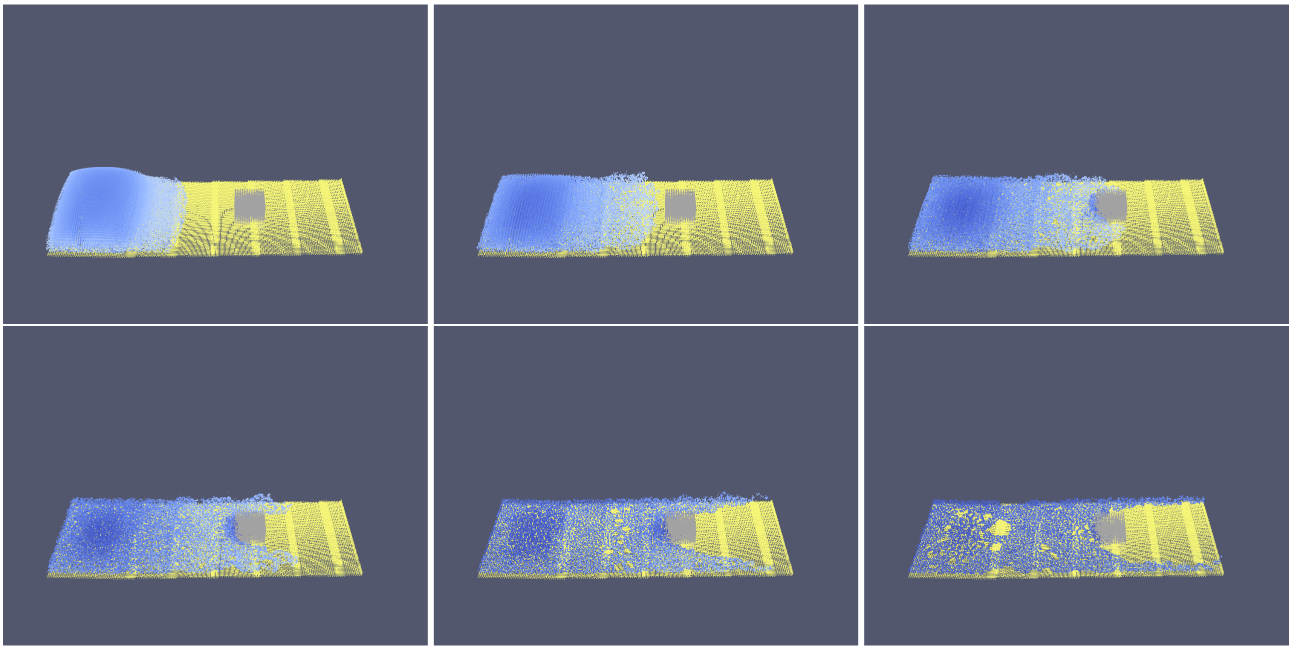


Figure 3.3: Wave Dynamics Simulation

3.4.7 Influence of Moisture Content

Another important factor that contributes to the stability of sandcastle is the moisture content of sand. Based on our SPH model discussed before, moisture content will mainly influence three parameters of the sand particle: density, cohesion coefficient, and internal friction angle. Once we have found the detailed relation between moisture content and these three parameters, we can substitute the relationship into the original model and analyze the proportion of potential energy loss again.

Zhang and his group [6] have applied experiment method to examine the relation between moisture content, dry density, cohesion pressure, and internal friction angle. Their experimental results are recorded in Appendices.

Based on their experiment data, Zhang and his group apply the method of polynomial fitting and obtain the relation function between these four parameters. We quote their empirical formula here:

$$\theta = 0.78 + 4.14w + 36.28\rho_d - 131.86w\rho_d \tag{21}$$

$$c = 219514w^3 + 89\rho_d^3 - 115980w^2 - 385\rho_d^2 + 20009w + 589\rho_d - 106w\rho_d - 1397 \tag{22}$$

where θ is the internal friction angle, c is the cohesion pressure, w is the moisture content, and ρ_d is the dry density of the sand.

Now we come to the relation between moisture content and true density of sand. Based on our assumption 3, the volume of sand will not change when we adding small amount of water into it. Hence the function is given by:

$$\rho_i = \rho_d(1 + w) \tag{23}$$

According to [12], the density of dry sand in natual world is typically $1.555g/cm^3$. Insert this value into Equation (18), (19), and (20), we can have the function describing relation between moisture content and internal friction angle, moisture content and cohesion coefficient, moisture content and true sand density.

Since Equation (18) and (19) are empirical formula based on the existing data, we should specify that the conditions for the use of these two formulas should be restricted. The value of w should between 12% and 22%. According to [13], the optimal moisture content should be around 12.25%, and hence our range is enough for computing.

Consequently, using a certain value of moisture content, we can calculate the results of internal friction angle, cohesion pressure, and wet density of sand particles. Inserting these three values in to our previous model, we can obtain the relation between proportion of potential energy loss and moisture content of sand. Hence we can find the best moisture content to build the most stable sandcastle.

3.4.8 Rainfall Simulation

Similar as Section 3.4.1, we apply SPH model to do the calculation. The difference is that the origin position of liquid particles is not on the same level as the sand to simulate waves and tides. Instead they were positioned in the air ten times as high as the sandcastle to simulate rainfall. We randomly choose twenty points at that altitude as the birth place of raindrops. Each drop consists of ten liquid particles. At the origin time, they were released without speed. Hence we could again utilize Algorithm 1 to do the simulation but simply change the input value of the liquid. Energy loss proportion ϵ_i is applied again to determine the damage extent of our designed geometric shape.

4 Results

4.1 3-Dimensional Geometric Shape Determination

We use UGNX to construct the model and import it into the compilation software. We mainly consider five geometric shapes, cube, triangular prism, triangular pyramid, cone, and our designed airfoil-like streamlined body, as shown in Figure 4.1 below

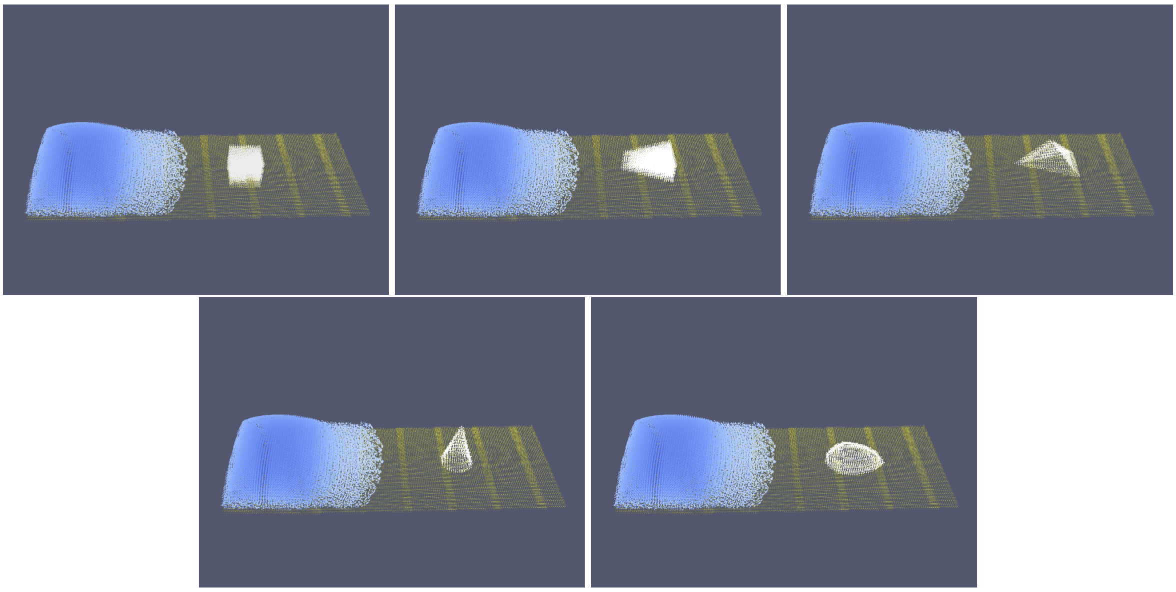


Figure 4.1: Simulation of Five Models

We generate waves and make them flow towards the foundation as we have discussed before. After each wave, we calculate the ratio of total potential energy and initial energy of the foundation and recorded in Figure 4.2 below:

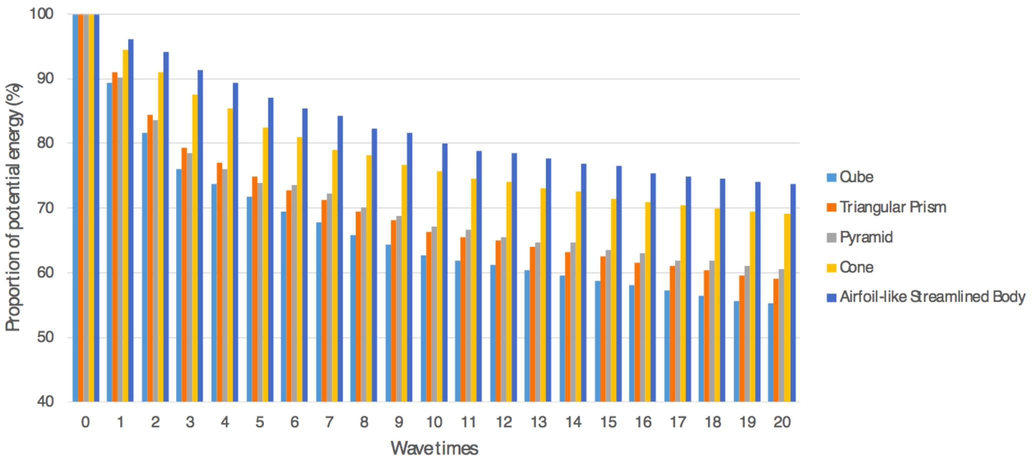


Figure 4.2: Proportion of Potential Energy vs. Wave Times

As we can see from the graph, the proportion of potential energy of each foundation falls down as the times of waves increase, indicating that the foundation is eroded by the waves. However, different shapes perform differently to waves. Cube perform the worst, since its proportion of energy decreases most sharply. Then comes pyramid and triangular prism whose energy proportion are in a similar decreasing rate. Above them is cone, which perform the second. Airfoil-like streamlined body is the one that works the best under the waves. Hence we conclude that our designed model is the best geometric foundation among these basic models.

4.2 Sand-to-Water Mixture Proportion Determination

To simplify the calculation, we choose several value of moisture content as feature points to do the calculation. Our calculation results are recorded below in Table 3:

Moisture Content	Internal Friction Angle	Cohesion Pressure	Wet Density
%	^{o}	kPa	$g * cm^{-3}$
12	33.0871	13.1065	1.7416
14	29.0691	29.9202	1.7727
16	25.0510	27.7065	1.8038
18	21.0330	17.0022	1.8349
20	17.0149	8.3439	1.8660
22	12.9969	12.2684	1.8971

Table 3: Value of Six Feature Points

Inserting these values into previous model and do the simulation, our results are recorded in Figure 4.3 below.

According to this graph, we find that with a moisture content of 14%, the foundation can suffer least damage from the waves and tides. Hence we claim that this is the best moisture content for sand particles. Based on the moisture content, we therefore calculate that the mass ratio and volume ratio of sand and water are roughly 7.1:1 and 4.6:1, respectively.

Consequently, we have also gained the best internal friction angle, cohesion pressure, and wet density of sand particles. We apply these values for other three problems in this paper, as we have discussed in Section 4.4.4.

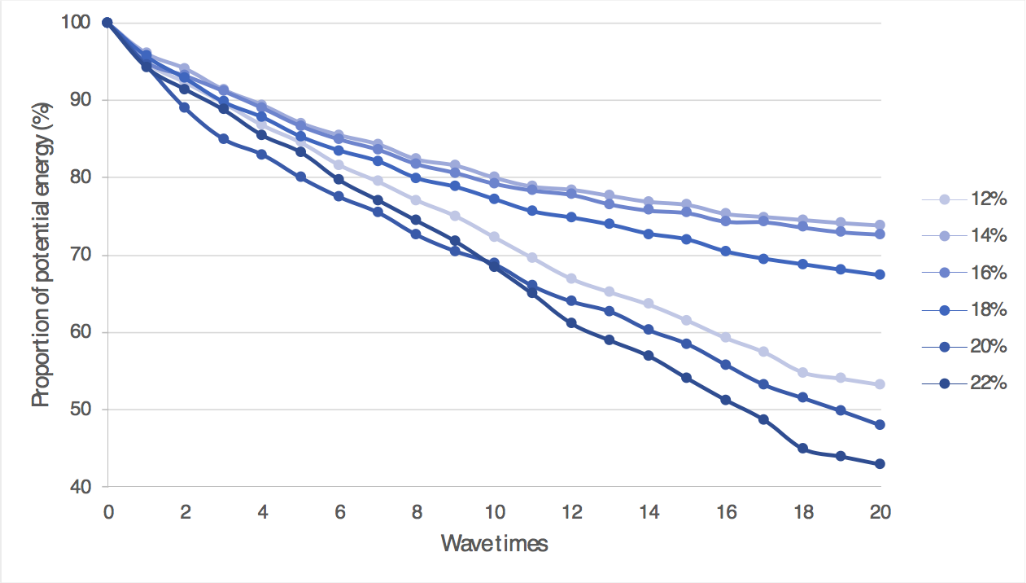


Figure 4.3: Moisture Content Influence

4.3 Influence of Rain Analysis

Similar as Section 5.1, we apply SPH again. This time the origin position of liquid is not on the left of the sand foundation, but over the sand instead. We still consider the five geometric shapes, cube, triangular prism, triangular pyramid, cone, and our designed airfoil-like streamlined body. We consider the precipitation capacity as $500ml/m^2 * h$, which is the amount of a moderate rain. The liquid particles are formed randomly above the foundation, as discussed before. Due to the complexity of the algorithm, the simulation speed is quite low, and it is difficult to simulate a time interval of one hour. Hence we have magnified rainfall per unit time by 60 times and apply the algorithm for 60 seconds to simulate the rainfall. Our simulation results are recorded below in Figure 4.4

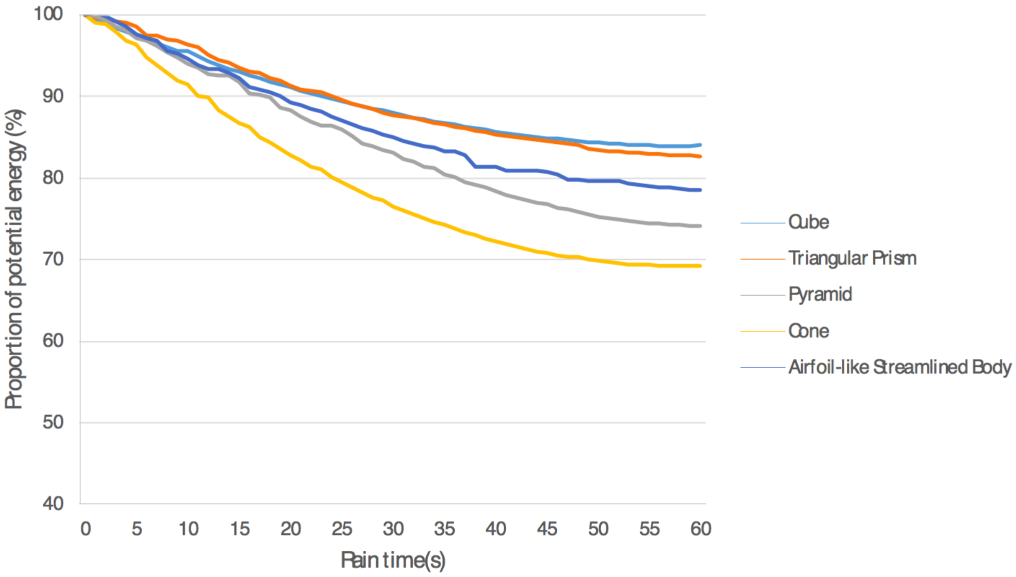


Figure 4.4: Proportion of Potential Energy vs. Time of Raining

According to the graph, we find some difference between what we get from Section 5.1. We find that our designed airfoil-like streamlined body does not perform the best in this situation, but ranked three instead. With a similar decreasing rate of proportion potential energy, cube and triangular prism become the best geometric shape under the influence of continuous raining. The results appear that there have been some changes in the rankings of the five geometric shapes. This phenomenon is easy to understand. The rain drops fall vertically from the air and strike the top of the foundation. With a horizontal top surface, cube and triangular prism are easy to withstand the raining. However, our designed airfoil-like streamlined body, as well as cone and pyramid do not possess a horizontal surface on their top. Sand particles are easily influenced by the rain particles and finally fall down from the slope, resulting in a decline of potential energy.

4.4 Further Strategies

In this section, we consider some other useful strategies for sandcastle-makers to build their stronger sandcastles. Digging a drainage ditch, building a wall, and choosing a steep position will be discussed in details.

4.4.1 Digging a Drainage Ditch

There are many ancient castles built with moats to protect them from the enemy. Facing the moat, enemy troops will find it hard to climb on the city wall. Hence we consider whether a "moat" could protect sandcastle better as well. As shown in Figure 4.5, We dug a drainage ditch all around the sandcastle. This ditch is designed to withstand the waves better. Figure 4.6 shows the synamic particle visualization of digging a drainage ditch. Using SPH model, we compare the proportion of potential energy loss of a normal streamlined body and a modified one with a drainage ditch. Our simulation results are shown in Figure 4.7 .

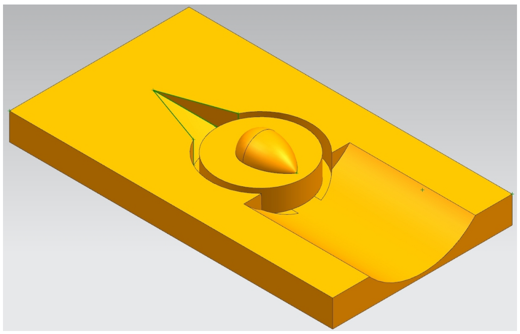


Figure 4.5: Drainage Ditch Model for Sandcastle

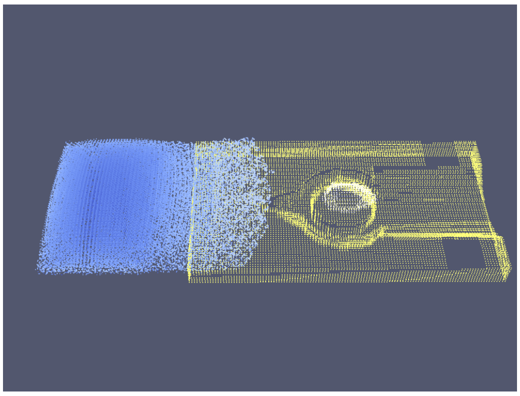


Figure 4.6: Dynamic Particle Visualization of Digging a Drainage Ditch

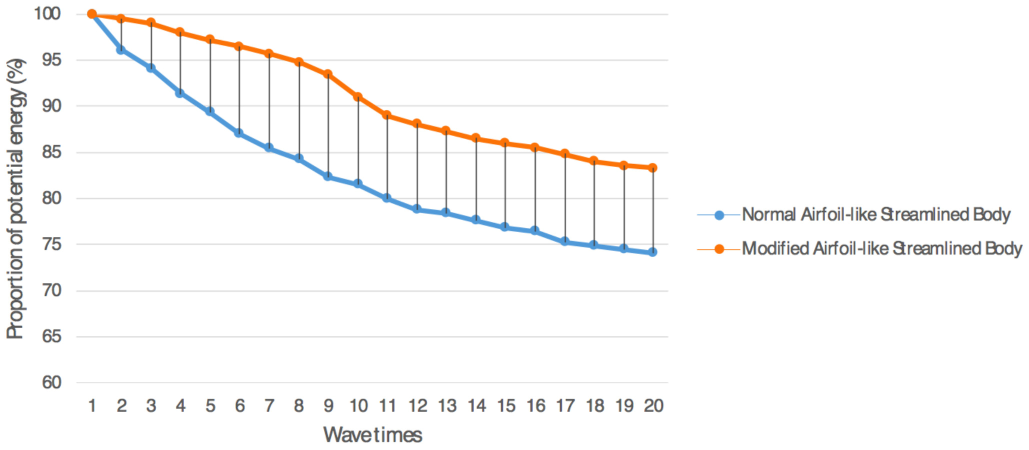


Figure 4.7: Comparison of Normal and Modified Streamlined Structure

According to Figure 4.7, digging a drainage ditch could decrease the energy loss to a great extent. At the first several waves, the energy decreasing rate of modified streamlined body is far less than that of the normal ones. However, when times of waves increases, this difference shinks and finally disappears. Potential energy of the two structures decreases almost at a same rate. This phenomenon is probably because that the drainage ditch is filled by the sea water during the later waves. Anyway, it cannot be denied that digging a ditch can help the sandcastle stand longer.

4.4.2 Building the Wall

Same as the effect of drainage ditch, we wonder whether a sandy wall will make a difference. Actually it does. The simulation procedures are similar to what we got in previous section. Due to time and space limits, we will not do the simulation here, but this will be covered in Possible Improvement part.

4.4.3 Choosing a Steep Position

It is obvious that building the sandcastle further up the beach in a more steep position will protect it much more effectively, and we apply our SPH model to specific the results. Our simulation results are posted below in Figure 4.8.

The outcoming theorem is quite direct. With a larger inclined angle, the foundation of the sandcastle will be less damaged by the waves and tides, since the fluid have to overcome the gavity force a lot. Consequently, building the sandcastle in a slope maybe a good strategy.

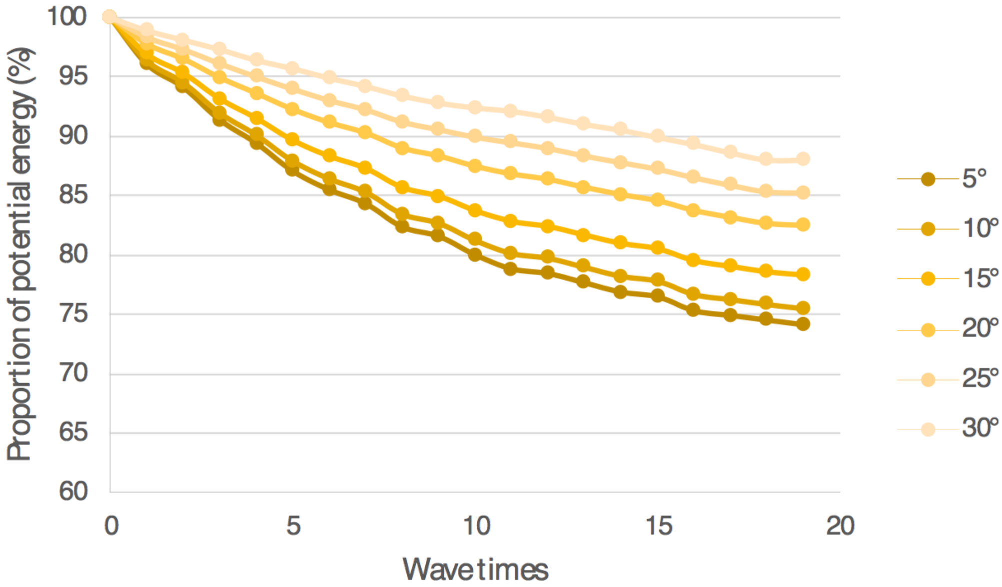


Figure 4.8: Proportion of Potential Energy Loss of Different Angles

4.4.4 Other Thinkings

We also consider some other strategies. A smooth or tough surface of the foundation may also influence its stability. Hence we suppose that castle-builders should make a smoother surface for their creation. Moreover, building the castle far away from the seashore is also a helpful way.

5 Discussion

5.1 Sensitivity Analysis

5.1.1 Parameters of Geometric Shape

One parameter that may influence the results of our calculation is the radius ratio κ. Again, we apply SPH model and keep all other parameters the same, but only change the radius ratio κ. We then examine how the airfoil-like streamlined body shaped foundation would perform when changing its radius ratio. Our simulation results are recorded below in Figure 5.1.

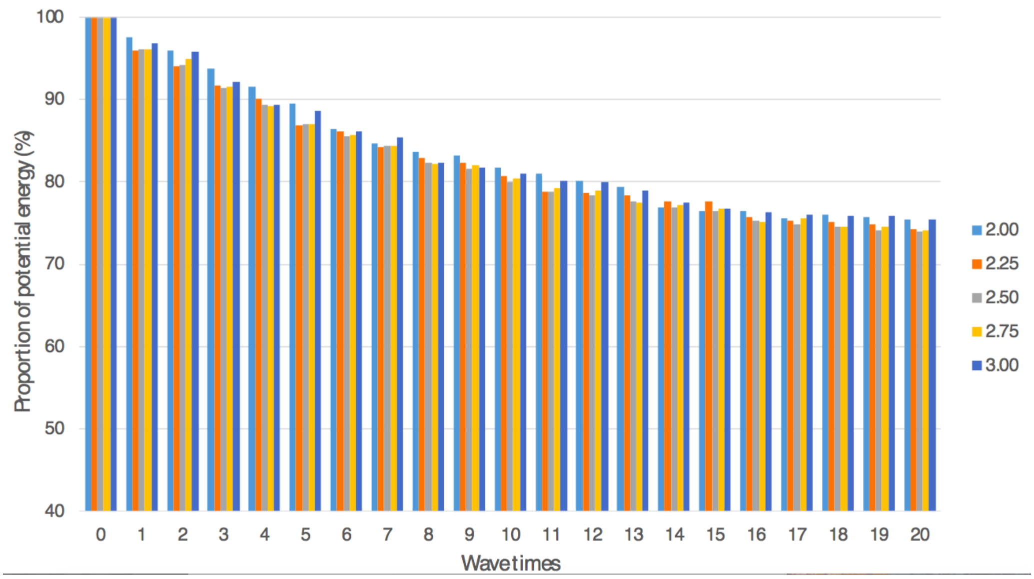


Figure 5.1: Sensitivity with Radius Ratio

Analyzing this figure, we find out that changing the radius radio wouldn't affect the stability of the foundation a lot. In order to quantify the difference, we calculate the bias of proportion of potential energy loss on 5, 10, 15, and 20 wave times, which are recorded below in Table 4

Wave Times	2.00	2.25	2.50	2.75	3.00
5	0.028418	0.001968	0	0.000579	0.017418
10	0.022345	0.009094	0	0.005573	0.012393
15	0.000639	0.014740	0	0.003297	0.003000
20	0.021349	0.009677	0	0.005159	0.022968

Table 4: Bias of ϵ in Different Radius Ratio

Hence we conclude that the radius ratio will not influence the stability of our designed model a lot. It is reasonable to use $\kappa = 2.50$ as our airfoil-like streamlined body's radius ratio.

5.1.2 Sensitivity of Polynomial Fitting

In section 4.4.7, we provide two polynomial fitting functions based on Zhang's article.6 The accuracy degree during the fitting influences the sensitivity of SPH simulation results a lot. Zhang provides a method of Range Analysis in 6, which calculates the absolute value of the difference value of maximum and minimum average. This value can indicate the order of each influential factor. Zhang apply the Range Analysis to the two parameters, internal friction angle and cohesion pressure. This method is very helpful to analyze the sensitivity of polynomial fitting, and we quad their work in Appendices.

5.2 Advantages and Limitations

Our SPH model treat fluid as a set of particles, and analyze the dynamic motion of them. It is a pure Lagrange method and can avoid the interface problem between materials and network that Euler method always encounters. Consequently, our model is extremely suitable for solving high velocity impact problems. In this question, we are asked to determine the interaction between liquid and sand. Hence SPH can be an optimal method to solving this problem.

Moreover, since every particle maintains its own physical characteristics, SPH is very suitable for multi-materials problems. We are not bothering about interface between different materials, since Porous flow simulation and Drucker-Prager model provides us with a very specific way to do the calculation.

However, during our analysis, we only consider five basic geometric shapes, and choose their best one. This consideration is lack of completeness and can not include all the shapes in mathematical world.

5.3 Possible Improvement

The accuracy of SPH simulation directly depends on the amount of particles we take into consideration. Generally speaking, the larger amount of particles, the more accuracy the result will be. Hence we can consider more liquid or sand particles to increase the accuracy of our system.

Secondly, more geometric shapes can be taken into consideration. As for the determined airfoil-like streamlined body, we can also think of some ways to improve its ability. One way we have considered is using spheroid to replace the standard round ball. This attempt can be fulfilled in the future works.

Moreover, as for Section 5.3, the simulation of rainfalls, we only consider the rain drops vertically. However, in actual situations, this is not always the case. Due to the influence of the wind, rain falls at an angle most of the time. Since our model is difficult to analyze the effect of wind, so it is impossible for us to simulate slant rain drops. However, this could be a great improvement and may be accomplished in the future. In that way, the results may be changed a lot.

Furthermore, we have considered many useful strategies for the fourth problem. Unfortunately, due to time limit, we cannot accomplish them all. Our strategies, including building a wall, making a smoother surface, and building our sandcastle faraway from the seashore, maybe simulated in the future.

5.4 Conclusion

During the whole procedure, we successfully applied SPH model to calculate the interaction between sand and liquid. We find out that our designed Airfoil-like streamlined body performs the best under the erosion of waves among other common 3-dimensional geometric shapes, including cube, triangular prism, triangular pyramid, and cone.

Moreover, when we mix sand and water together, the best moisture content for the mixture is 14%. That is, the optimal volume ratio to stable the sandcastle is approximately 7:1, and the mass ratio is around 5:1.

Thirdly, airfoil-like streamlined body does not perform the best among these shapes when it is affected by the rain. Instead, cube and triangular prism become the best geometric shape due to their horizontal top surface. Hence we calim that with a horizontal top, a foundation can withstand the rainfall best.

Finally, we put forward three strategies to help sandcastle last longer. Digging a drainage ditch, building a wall, and building the sandcastle at a steep position can help a lot.

In the next section, we will provide an informative article describing our discovery through the whole procedure. This article is writing for non-technical readers of a vacation magazine, *Fun in the Sun*.

6 Article for *Fun in the Sun*

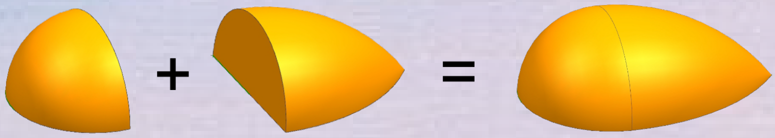
TAKE A TIME TO BUILD YOUR SANDCASTLE!

Overwhelmed by the burdens of work and study? Tired of the hustle and bustle of life in a big city? How about take a break from the normal life and spend an afternoon enjoying the sunshine on the coast!

All you need is just a shovel and a bucket, and a heart of imagination. When all of the tools are prepared, you are able to begin your creation.

Before building a beach castle, one thing you should remember is to find a proper place to build your sandcastle. A slope slightly away from the coast with enough soft sand is an ideal selection. Another important thing is wet the sand with water. An crucial factor determining how long your sandcastle can maintain under the assault of the seawater is the ratio of water and sand combined together. Without the help of water, sand cannot stably gather together. Mix the sand with the optimal amount of water can make them sticker. However, too much water is also not a good choice, since sand will flow away together with the water. It's as easy as clay to make the shape you want. The optimal mass proportion of sand and water is roughly 7:1, and volume proportion is approximately 5:1. That is, you need to mix five buckets of sand with one bucket of water together. Your sand will be stable as clay if you do it this way.

After you have prepared the wetted sand, the next step is to build the foundation of your sandcastle. Different shapes of foundation will perform differently to the waves and tides? Don't know what shape is the best? Let me tell you! Simulation results give out that the airfoil-like streamlined body can be the most effective foundation. Imagine what an engine of a airplane looks like. Cut the shape in half from the central plane and place it on the ground, that is exactly what a airfoil-like streamlined body looks like. One simply way to build this streamlined structure is as follows. First, build a quarter sand ball on the ground, with its smooth surface parallel to the coastline. Second, use sand to build a slightly slower slope behind the quarter ball. Remember to keep the surface of the foundation smooth and plain. You can make a reference to figure below. Easy, isn't it?



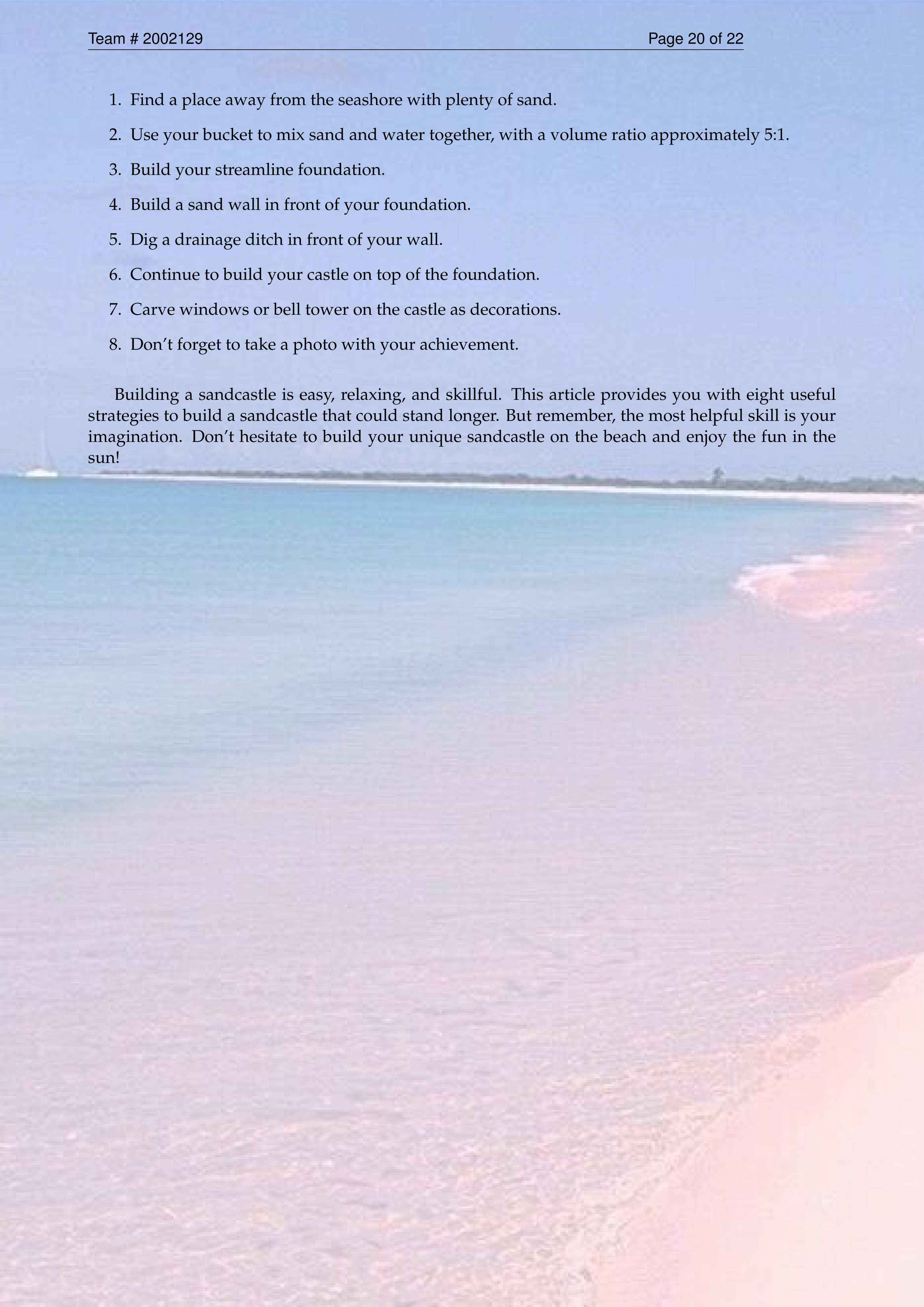
Try to recall what a fairytale castle looks like. Walls and moats are standard features of almost every castle. They can prevent the invade of armies and decorate the castle. This is same for sandcastles. Dig a moat about half a meter in front of the castle. It doesn't need to be very deep. Ten centimeters deep is enough, but it needs to be as wide as your castle. Then build a ten-centimeter-high sand wall just behind the moat. This wall is very effective against the invasion of the waves and tides. If you have plenty of time, you can even have moats and walls all around your castle. They are not only protective, but also beautiful.

When you have completed the foundation of the sandcastle, you can decorate your castle with your imagination as much as you like. Windows, balconies, bell towers, even bridges. As long as you keep a proper proportion of sand and water, you can create anything you imagine just using these simply materials. Don't believe me? Just try it!

After the above introduction, I believe you can't wait to try it out for yourself. A list is provided for you to follow when constructing your own sandcastle:

1. Find a place away from the seashore with plenty of sand.
2. Use your bucket to mix sand and water together, with a volume ratio approximately 5:1.
3. Build your streamline foundation.
4. Build a sand wall in front of your foundation.
5. Dig a drainage ditch in front of your wall.
6. Continue to build your castle on top of the foundation.
7. Carve windows or bell tower on the castle as decorations.
8. Don't forget to take a photo with your achievement.

Building a sandcastle is easy, relaxing, and skillful. This article provides you with eight useful strategies to build a sandcastle that could stand longer. But remember, the most helpful skill is your imagination. Don't hesitate to build your unique sandcastle on the beach and enjoy the fun in the sun!



References

[1] Wei-Chun Teng: *Particle-Based Fluid Simulation for Sand Sculpture and Water*. National Chiao Tung University.

[2] Muller M, Charypar D, Gross M: *Particle-based fluid simulation for interactive applications*. Eurographics Association, pp. 154–159.

[3] Monaghan J.: *Smoothed particle hydrodynamics*. Annual Revision on Astronomy and Astrophysics 30 (1992), 543–574.

[4] Zhu Y., Bridson R.: *Animating sand as a fluid*. In SIGGRAPH '05: ACM SIGGRAPH 2005 Papers (New York, NY, USA, 2005), ACM Press, pp. 965–972.

[5] Al-Shayea, N, A. *The combined effect of clay and moisture content on the behavior of remolded unsaturated soils[J]*. ENGINEERING GEOLOGY -AMSTERDAM-, 2001.

[6] Zhang Kui, Li Meng-zi, Yang Bei-bei: *Research on Effect of Water Content and Dry Density on Shear Strength of Remolded Loess*. Journal of Anhui University of Science and Technology(Natural Science), 2016.

[7] Rüdiger Schwarze, Anton Gladkyy, Fabian Uhlig, etc. *Rheology of weakly wetted granular materials: a comparison of experimental and numerical data[J]*. Granular Matter, 15(4):455-465.

[8] https://blog.csdn.net/liuyunduo/article/details/84098884

[9] Lenaerts T, Adams B, Dutré, Philip:*Porous flow in particle-based fluid simulations[J]*. Acm Transactions on Graphics, 2008, 27(3):1-8.

[10] http://github.com/DualSPHysics/DualSPHysics/wiki/3.-SPH-formulation#39-wave-generation

[11] Darcy, H. 1856. Les Fontaines Publiques de la Ville de Dijon, Dalmont, Paris.

[12] https://www.engineeringtoolbox.com/dirt-mud-densities-d_1727.html

[13] https://www.scientificamerican.com/article/how-to-build-the-perfect-sandcastle-according-to-science/

Appendices

Appendix A Experimental Results for Sand[6]

Moisture Content	**Dry Density**	**Cohesion Pressure**	**Internal Friction Angle**	$100kPa$**shear strength**
$\%$	$g * cm^{-3}$	kPa	^{o}	kPa
12.87	1.4	21.08	27.11	72.2
12.87	1.5	21.64	30.50	85.0
12.87	1.6	23.92	33.50	99.2
12.87	1.7	26.82	34.98	112.8
15.78	1.4	24.96	24.48	70.5
15.78	1.5	27.68	25.59	75.5
15.78	1.6	29.20	25.87	77.7
15.78	1.7	32.64	26.01	75.2
18.31	1.4	12.80	17.95	77.0
18.31	1.5	13.60	18.42	48.0
18.31	1.6	16.02	20.11	52.6
18.31	1.7	19.40	21.11	58.0
21.50	1.4	8.00	12.63	30.4
21.50	1.5	9.80	14.07	34.8
21.50	1.6	10.62	15.11	37.6
21.50	1.7	11.23	16.32	40.5

Table 5: Experimental Results for Moisture Content, Dry Density, and Cohesion Pressure

Appendix B Range Analysis[6]

Table 6: Range Analysis for Internal Friction Angle

	Moisture Content	**Dry Density**
$\overline{I}$	31.54	20.54
$\overline{II}$	25.49	22.15
$\overline{III}$	19.40	23.67
$\overline{IV}$	14.53	24.61
Range R	17.01	4.07

Table 7: Range Analysis for Cohesion Pressure

	Moisture Content	**Dry Density**
$\overline{I}$	23.37	16.71
$\overline{II}$	28.62	18.18
$\overline{III}$	15.46	19.94
$\overline{IV}$	9.91	22.52
Range R	18.71	5.81